Looking For All Solutions of a Set of Max-Atoms Solves the Max Atom Problem in Strongly Polynomial Time

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Abstract

In this paper we consider a set of max-atoms (SMA) that is a set of conjunctions of m inequalities of the form $x_i \leq a + max(x_j, x_k)$ with offset $a \in \mathbb{R}$ and x_i, x_j, x_k are the variables elements of the whole set $V = \{x_1, \ldots, x_n\}$. Considering the one-to-one correspondance of max-atoms and offsets the number of elements of the possibly multiset of offsets is m. The SMA has always the vector of variables $x = (-\infty)$ as trivial solution. On this object we study two questions. MAP: can we find at least one non trivial solution to the SMA in strongly polynomial time ? and MAPall: can we find all the solutions of the SMA in strongly polynomial time ?
We distinguish two cases w.r.t the hypothesis **(W)**: there exists at least one strictly negative max-atom, ie an inequality $x_i \leq a + max(x_j, x_k)$ with $a < 0$.

When **(W)** is false there exists at least the non trivial solution $x = (0)$ to the SMA, indeed $0 \leq a + \max(0, 0)$ $\forall a \geq 0$. And MAP is yes.
When **(W)** is true we prove in this paper that MAPall is yes with complexity $\mathcal{O}(mn^3)$. And thus MAP is also yes. So in all cases MAP is yes with time complexity $\max(\mathcal{O}(1), \mathcal{O}(mn^3)) = \mathcal{O}(mn^3))$.

The important consequence of this result is that the following six PTIME equivalent problems to MAP which are known to be in NP intersection co-NP are also strongly polynonial. P1: Looking for non trivial solutions of a tropical cone, P2: Computation of a tropical rank of a matrix, P3: Computation of optimal strategies in parity games (typically: Mean Payoff Games), P4: Scheduling with and/or precedence constraints, P5: Shortest path problem in hypergraph, P6: Model checking and μ-calculus.

The whole method developed in this paper is illustrated on an example of SMA which is PTIME equivalent to P5.

1 Introduction

A max-atom of offset $a \in \mathbb{R}$ is an inequality of the form $x_i \leq a + \max(x_j, x_k)$ (eventually we could have $j = k$, ie $x_i \leq a + x_j$) where x_i, x_j, x_k are the variables elements of the whole set $V = \{x_1, \ldots, x_n\}$. This set is stored in the n-dimensional column vector $x = (x_i)_{i=1}^n$. Because $a + \max(x_j, x_k) = \max(a + x_j, a + x_k)$ a max-atom is a $(\max, +)$-linear way to express the disjunction $(x_i \leq a + x_j)$ or $(x_i \leq a + x_k)$. In this paper we consider a set of m max-atoms (SMA) with m offsets. The SMA will be denoted $\mathcal{A}$ in the whole theoretical part of the paper. And we are studying solutions in $\mathbb{R} \cup \{-\infty\}$. The trivial solution of the SMA is obviously $x = (-\infty)$. Due to the disjunction equivalence we have the following obvious PRETREATMENT: $x_i \leq a + \max(x_i, x_k)$ with $a \geq 0$ is always satisfied because $x_i \leq a + x_i$ is true $\forall x_i \in \mathbb{R} \cup \{-\infty\}$. Thus, we remove it. And the inequality $x_i \leq a + \max(x_i, x_k)$ with $a < 0$ is equivalent to $x_i \leq a + x_k$ (indeed, the inequality $x_i \leq a + x_i$ is false $\forall x_i \in \mathbb{R}$).

On the SMA we are studying two questions. MAP: can we find at least one non trivial solution to the SMA in strongly polynomial time ? and MAPall: can we find all the solutions

of the SMA in strongly polynomial time ?
We distinguish two cases w.r.t the hypothesis **(W)**: there exists at least one strictly negative max atom, ie an inequality $x_i \leq a + max(x_j, x_k)$ with $a < 0$.

The **main result is that MAP is solved in strongly polynomial time**. Indeed, when **(W)** is false there exists at least the non trivial solution $x = (0)$ to the SMA, because obviously we have: $0 \leq a + \max(0, 0)$ $\forall a \geq 0$. And MAP is yes.
When **(W)** is true we prove in this paper that MAPall is yes. And thus MAP is also yes. So in all cases MAP is yes.

And **the main consequence is that the six problems Pk, $k = 1, \ldots, 6$ mentioned in the abstract are also strongly polynomial**. These problems were known to be in NP $\cap$ co-NP (see eg [1], [3], [5], [6], [7], [8], [9], [10], [11], [13]).

In this paper we prove that MAPall is yes in the case where **(W)** is assumed to be true. The question MAPall in the case where **(W)** is assumed to be false is studied in the working paper [12]. The results of this paper are illustrated on SMA PTIME equivalent to the hypergraph problem (ie problem P5 mentioned in the abstract).

The **TWO MAIN IDEAS** of the paper are (1) to use (max, +)-linear formalism and (2) define and characterize the notion of free variable of SMA (see Definition 4.1, Propositions 4.1 and 5.1).

1.1 SMA examples of hypergraph problem P5

To illustrate our theoretical results we will consider the SMA $S = \{x_3 \leq (-10) + x_1, x_4 \leq (-1) + \max(x_2, x_3), x_2 \leq x_4, x_4 \leq 25 + \max(x_2, x_3)\}$ borrowed from [4], Example 1 with the following change of notations $x_1 \leftrightarrow u$, $x_2 \leftrightarrow y$, $x_3 \leftrightarrow x$, $x_4 \leftrightarrow z$. In [4] it is proved that (SMA + MAP) is PTIME equivalent to the hypergraph problem P5. In particular, the SMA S + MAP corresponds to the hypergraph problem P5 of Figure 1. On S we only retrieve the result of [4]: MAP is yes. We also study the SMA $S' = \{x_3 \leq (-10) + x_1, x_4 \leq (-1) + \max(x_2, x_3), x_2 \leq x_4, x_4 \leq 25 + \max(x_2, x_3), x_1 \leq 9 + \max(x_2, x_3)\}$ borrowed from [4], Example 2. As in [4] we obtain that $x = (-\infty)$ is the only solution. Finally, we are looking for all solutions to the SMA $S'' = (S' \setminus \{x_1 \leq 9 + \max(x_2, x_3)\}) \cup \{x_1 \leq 12 + \max(x_2, x_3)\}$.

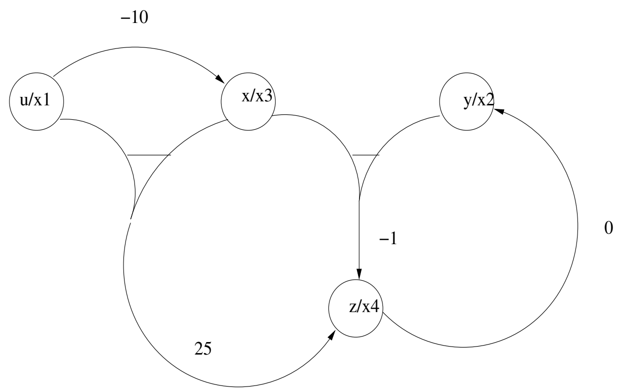

Figure 1: Hypergraph PTIME equivalent to the SMA S (see [4], Example 4, Fig. 1(a))

2 Useful results on $(\max, +)$-algebra

The reader is invinted to read eg [2] to find more details on $(\max, +)$-algebra and idempotent semi-rings. In what follows we only present the necessary and sufficient material to solve SMA + MAPall under assumption **(W)**.

The $(\max, +)$-algebra denotes the set $\mathbb{R}_{\mathbb{0}} := \mathbb{R} \cup \{-\infty\}$ equipped the "addition": $(a, b) \mapsto \max(a, b)$ and the "multiplication" $(a, b) \mapsto a + b$.
In the sequel we will use the following notations: $a \oplus b := \max(a, b)$, $a \otimes b := a + b$ (sometimes $a \otimes b$ could be denoted ab). $\mathbb{1} := 0$ is the neutral element for $\otimes$, $\mathbb{0} := -\infty$ is the neutral element for $\oplus$, $\mathbb{0}$ is also the absorbing element of $\otimes$. We will use the following power notation:

$$a^{\otimes(b)} := a \times b \text{ } (\times\text{: usual multiplication}).$$

Note that in particular we have: $a^{\otimes(0)} = \mathbb{1}$.
$\mathbb{R}_{\mathbb{0}}$ is equipped with the natural order $\leq$ defined by:

$$a \leq b \Leftrightarrow a \oplus b = b.$$

The minimum operator min will be denoted in the sequel: $\wedge$.
The scalar operations are generalized to matrices as follows:

- addition of two matrices: $(a_{i,j}) \oplus (b_{i,j}) := (a_{i,j} \oplus b_{i,j})$
- minimum of two matrices: $(a_{i,j}) \wedge (b_{i,j}) := (a_{i,j} \wedge b_{i,j})$
- product of two matrices: the entry (i, j) of the matrix $C = A \otimes B$ is defined by: $c_{i,j} := \oplus_k a_{i,k} \otimes b_{k,j}$ $(=\max_k(a_{i,k} + b_{k,j}))$.
- comparaison of two matrices: $A \leq B$ means $\forall i, j : a_{i,j} \leq b_{i,j}$.

The matrix I_n denotes the $n \times n$-identity matrix. Its diagonal entries are all $\mathbb{1}$ and its off-diagonal entries are all $\mathbb{0}$. And $O_{m,n}$ will denote the $m \times n$-null matrix (ie, all entries are $\mathbb{0}$).

In the sequel we will need resilts dealing with pseudo-inverse of matrix or residuation theory for matrix. Let m, n, q be three integers ≥ 1. We consider two matrices A: $m \times n$ and B: $m \times q$. Residuation theory allows us to find $\oplus\{X : A \otimes X \leq B\}$ in the sense of the partial order defined on matrices. Thus, we have to solve:

$$A \otimes X \leq B. \tag{1}$$

Using the fact that $a \oplus b \leq c \Leftrightarrow (a \leq c)$ and $(b \leq c)$ one can rewrite (1) as:

$$X \leq A \backslash B. \tag{2}$$

And the $n \times m$-matrix $A\backslash = [(a\backslash)_{i,j}]$ can be seen as a kind of pseudo-inverse matrix such that its entries $(a\backslash)_{i,j}$ are defined by:

$$\forall i = 1, \ldots, n, \forall j = 1, \ldots, m, (a\backslash)_{i,j} := a_{j,i}^{\otimes(-1)}, \tag{3}$$

with the convention $\mathbb{0}^{\otimes(-1)} := +\infty$ such that the one-to-one function $(\cdot)^{\otimes(-1)}$ is defined from $\mathbb{R} \cup \{-\infty\}$ onto $\mathbb{R} \cup \{+\infty\}$. The $n \times q$-matrix $A\backslash B$ is defined as the product of the $n \times m$-matrix $A\backslash$ with the $m \times q$-matrix B in the algebraic structure $(\mathbb{R}_{\mathbb{0}} \cup \{+\infty\}, \min := \wedge, + := \otimes')$ where $\wedge$ is considered as another "addition" with neutral element $+\infty$ and $\otimes'$ is another "multiplication" with neutral element $\mathbb{1}$ and absorbing element $+\infty$. In particular, we have the important equality: $\mathbb{0} \otimes' +\infty = +\infty$. In this new algebraic structure the entry (i, j) is defined by:

$$(A \backslash B)_{i,j} := \wedge_{k=1,\ldots,m}((a\backslash)_{i,k} \otimes' b_{k,i}). \tag{4}$$

The reader must be aware that different equivalent forms of the expression (4) can be provided notably because eg: $\min(a, b) = -\max(-a, -b)$.

To any $n \times n$-matrix A we can associate a valued graph $G(A)$ with valuation v such that the set of the vertices of $G(A)$ is $\{1, \ldots, n\}$. The valuation is defined by $v_{i,j} = a_{j,i}$ for all vertices i, j. We say that an arc $i \to j := (i, j)$ exists if $v_{i,j} \neq \mathbb{0}$. A path p of lenght $k \geq 1$ is a series of k consecutive arcs which has the form: $p = i_0 \to i_1 \to \cdots i_{k-1} \to i_k$. And the weight of p is defined by: $w(p) := v_{i_0,i_1} \otimes \cdots \otimes v_{i_{k-1},i_k} = a_{i_1,i_0} \otimes \cdots \otimes a_{i_k,i_{k-1}}$. An elementary circuit of lenght $k \geq 1$ is a special path such $\forall l = 0, \ldots, k-1$: $i_l \neq i_{l+1}$ and $i_k = i_0$.

Let A be a $n \times n$-matrix, b a n-dimensional column vector and $x = (x_i)_{i=1}^n$. We have the following well-known result.

Result 2.1 (Saturation). *Let us consider the set defined by:*

$$\mathcal{S}(A, b) := \{x : x \leq A \otimes x \oplus b\}. \tag{5}$$

If the following condition is verified:

$$\exists \lim_{k \to +\infty} A^{\otimes(k)} = O \ (O\text{: null matrix}) \tag{6}$$

then the greatest element of the set $\mathcal{S}(A, b)$ is:

$$x = A^* \otimes b$$

where $A^* := I \oplus A \oplus A^{\otimes(2)} \oplus A^{\otimes(3)} \oplus \cdots$ is the infinite "sum" of the powers of the matrix A known as the Kleene star of the matrix A. And under condition (6) we have: $A^* = I \oplus A \oplus A^{\otimes(2)} \oplus \cdots \oplus A^{\otimes(n-1)}$.

This greatest element is the solution of the $(\max, +)$-linear equation: $x = A \otimes x \oplus b$ which is obtained by the saturation of the inequality: $x \leq A \otimes x \oplus b$.

There exist different conditions such that (6) is true. Notably the one which states that all elementary circuits of the valued graph $G(A)$ associated with the matrix A have weight $< \mathbb{1}$. But here our guideline will be the following sufficient condition:

$$A = (a_{i,j}) \text{ s. t } : \forall i, j \ a_{i,j} < \mathbb{1}. \tag{7}$$

It is clear that (7) implies that all elementary circuits of the valued graph $G(A)$ associated with the matrix A have weight $< \mathbb{1}$ which implies that $\exists \lim_{k \to +\infty} A^{\otimes(k)} = O$.

3 Matrix form of the SMA

The main idea of this section is to put the SMA $\mathcal{A}$ in the following matrix form:

$$x \leq A_1 \otimes x \wedge \ldots \wedge A_\ell \otimes x \wedge A_{\ell+1} \otimes x \ldots \wedge A_L \otimes x. \tag{8}$$

Where $[A_1, \ldots, A_\ell]$ are $n \times n$-matrices which verify condition (7) called strictly negative part of $\mathcal{A}$, say $\mathcal{A}_-$, and $[A_{\ell+1}, \ldots, A_L]$ are $n \times n$-matrices which **do not** verify condition (7) called positive part of $\mathcal{A}$, say A_+. The integers $\ell \geq 1$ and $L \geq \ell + 1$ will be defined in subsection 3.2.

3.1 Matrix form of the SMA S, subsection 1.1

Using the $(\max,+)$-linearity we have $S = \{x_3 \leq (-10) + x_1, x_4 \leq (-1) + \max(x_2, x_3), x_2 \leq x_4, x_4 \leq 25 + \max(x_2, x_3)\} = \{x_3 \leq 10^{\otimes(-1)} \otimes x_1, x_4 \leq 1^{\otimes(-1)} \otimes (x_2 \oplus x_3), x_2 \leq \mathbb{1} \otimes x_4, x_4 \leq 25 \otimes (x_2 \oplus x_3)\}$. Thus, by adding the inequality always true $x_1 \leq \mathbb{1} \otimes x_1$ the SMA S can be rewritten as follows:

$$\begin{array}{rcc} x_1 & \leq & \mathbb{1} \otimes x_1 \\ x_2 & \leq & \mathbb{1} \otimes x_4 \\ x_3 & \leq & 10^{\otimes(-1)} \otimes x_1 \\ x_4 & \leq & (1^{\otimes(-1)} \otimes (x_2 \oplus x_3)) \wedge (25 \otimes (x_2 \oplus x_3)). \end{array}$$

The condition (7) provides the following partition of S. The strictly negative SMA $S_- = \{x_3 \leq 10^{\otimes(-1)} \otimes x_1, 1^{\otimes(-1)} \otimes (x_2 \oplus x_3)\}$ logically equivalent to $S_- = \{x_1 \leq \mathbb{1} \otimes x_1, x_2 \otimes \mathbb{1} \otimes, x_3 \leq 10^{\otimes(-1)} \otimes x_1, 1^{\otimes(-1)} \otimes (x_2 \oplus x_3)\}$. And the positive SMA $S_+ = \{x_2 \leq \mathbb{1} \otimes x_4, x_4 \leq 25 \otimes (x_2 \oplus x_3)\}$ logically equivalent to $S_+ = \{x_1 \leq \mathbb{1} \otimes x_1, x_2 \leq \mathbb{1} \otimes x_4, x_3 \leq \mathbb{1} \otimes x_3, x_4 \leq 25 \otimes (x_2 \oplus x_3)\}$.

The desired matrix form (8) of S is:

$$x \leq \underbrace{A_1 \otimes x}_{S_-} \wedge \underbrace{A_2 \otimes x}_{S_+}.$$

Where

$$A_1 := \begin{pmatrix} \mathbb{1} & \mathbb{0} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{1} & \mathbb{0} & \mathbb{0} \\ 10^{\otimes(-1)} & \mathbb{0} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & 1^{\otimes(-1)} & 1^{\otimes(-1)} & \mathbb{0} \end{pmatrix}, \; A_2 := \begin{pmatrix} \mathbb{1} & \mathbb{0} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & \mathbb{1} \\ \mathbb{0} & \mathbb{0} & \mathbb{1} & \mathbb{0} \\ \mathbb{0} & 25 & 25 & \mathbb{0} \end{pmatrix}. \tag{9}$$

With the integers $\ell = 1$ and $L = 2$.

3.2 General case

In this subsection we explain how to obtain the inequality (8) logically equivalent to the SMA $\mathcal{A}$.

The n-dimensional row vectors of I_n form the canonic basis of $\mathbb{R}^n_{\mathbb{0}}$ and will be denoted $e_1, \ldots, e_n$. To any max-atom of $\mathcal{A}$, typically an inequality $x_i \leq a + \max(x_j, x_k)$, one can associate its $(\max,+)$-linear form, say $x_i \leq a \otimes (x_j \oplus x_k)$. And this $(\max,+)$-linear inequality can be rewritten as $x_i \leq a_i \otimes x$ where the n-dimensional row vector $a_i := a \otimes e_j \oplus a \otimes e_k$. And clearly, any max-atom can be identified with a n-dimensional row vector of the same kind than vector a_i. These vectors will be called *a_i-type vectors*. In the sequel we only deal with these a_i-type vectors.

For each variable x_i of $\mathcal{A}$, $i = 1, \ldots, n$ we associate:

- the list of strictly negative max-atoms $\mathcal{L}^-(x_i)$. And if this list is empty we add the *trivial max-atom* $\{x_i \leq x_i\}$ which corresponds to the a_i-type vector e_i,
- the list of the positive max-atoms $\mathcal{L}^+(x_i)$. And if this list is empty we add the trivial max-atom $\{x_i \leq x_i\}$ which corresponds to the a_i-type vector e_i.

We define the integers:

$$\ell := \max_{i=1,\ldots,n} (\mathsf{Cardinal}(\mathcal{L}^-(x_i))), \tag{10a}$$

and

$$L := \max_{i=1,\dots,n} (\mathsf{Cardinal}(\mathcal{L}^+(x_i)) + \ell. \tag{10b}$$

And because the LOGICAL AND is idempotent we can repeat any max-atom as many times as needed so that we can assume w. l. o. g. that $\forall x_i$: $\mathsf{Cardinal}(\mathcal{L}^-(x_i)) = \ell$ and $\mathsf{Cardinal}(\mathcal{L}^+(x_i)) = L - \ell$ where $\mathcal{L}^-(x_i)$ and $\mathcal{L}^+(x_i)$ are now considered as multisets (possibly there exist repeated elements). Thus, we have: $\mathcal{L}^-(x_i) = \{\{a_i^1, \dots a_i^\ell\}\}$ and $\mathcal{L}^+(x_i) = \{\{a_i^{\ell+1}, \dots, a_i^L\}\}$ where the vectors $a_i^l, l = 1, \dots, L$ are a_i-type vectors.

And $\mathcal{A}$ is equivalent to the inequality (8) where the $n \times n$-matrices A_l, $l = 1, \dots, L$ are defined by:

$$A_l := \begin{pmatrix} a_1^l \\ \vdots \\ a_n^l \end{pmatrix}. \tag{11}$$

Note that **(W)** $\Rightarrow A_1 \neq I_n$. Recall that because of condition (7) the SMA $\mathcal{A}$ is partitioned into two SMA: the non-empty strictly negative part $\mathcal{A}_-$ associated with matrices $A_l, l = 1, \dots, \ell$ and the positive part $\mathcal{A}_+$ associated with matrices $A_l, l = \ell + 1, \dots, L$.

4 How to use Result 2.1 to express all the solutions of $\mathcal{A}_-$

4.1 Explanations on a simple example

We consider the very simple strictly negative SMA with three variables $V := \{x_1, x_2, x_3\}$, $\mathcal{A} = \{x_3 \leq a \otimes (x_1 \oplus x_2)\}$, $a < \mathbb{1}$. The multisets associated with the variables are: $\mathcal{L}^-(x_1) = \{\{a_1^1 := e_1\}\}$, $\mathcal{L}^-(x_2) = \{\{a_2^1 := e_2\}\}$, $\mathcal{L}^-(x_1) = \{\{a_3^1 := a \otimes e_1 \oplus a \otimes e_2\}\}$ and $\mathcal{L}^+(x_1) = \{\{a_1^2 := e_1\}\}$, $\mathcal{L}^+(x_2) = \{\{a_2^2 := e_2\}\}$, $\mathcal{L}^+(x_2) = \{\{a_3^2 := e_3\}\}$. The integers $\ell = 1$ (see (10a)) and $L = 2$ (see (10b)). The matrix form (8) of $\mathcal{A}$ is $x \leq A_1 \otimes x \wedge A_2 \otimes x$, with $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $A_1 = \begin{pmatrix} I_2 & O_{2,1} \\ B & C \end{pmatrix}$. With $B = \begin{pmatrix} a & a \end{pmatrix}$, $C = \begin{pmatrix} \mathbb{0} \end{pmatrix}$ and $A_2 = I_3$. The partition $((x_1, x_2), x_3)$ of the matrix A_1 anticipates the fact that x_1, x_2 can take any arbitrary values in $\mathbb{R}_{\mathbb{0}}$ and thus in the sequel any variable which can take any arbitrary values in $\mathbb{R}_{\mathbb{0}}$ will be called *free variable*. On the contrary the variable x_3 is upper bounded by a (max, +)-linear function of x_1, x_2. This suggests to use Result 2.1 to express x as a function of $u := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. To do this we use the following decomposition: $A_1 \otimes x = A \otimes x \oplus b$, with $A = \begin{pmatrix} O_{2,2} & O_{2,1} \\ O_{1,2} & C \end{pmatrix}$, and $b = \begin{pmatrix} I_2 \\ B \end{pmatrix} \otimes u$.

Matrix A obviously verifies condition (6) and $A^* = \begin{pmatrix} I_2 & O_{2,1} \\ O_{1,2} & C^* \end{pmatrix}$, that is $A^* = I_3$ in our special case. Result 2.1 applies and the set $\mathcal{S}(A, b)$ has the greatest element $A^* \otimes b = \begin{pmatrix} I_2 \\ C^* \otimes B \end{pmatrix} \otimes u$. Let us denote $J = C^* \otimes B$ and $T = \begin{pmatrix} I_2 \\ J \end{pmatrix}$. Then, we have established the following set equality: $\{x : x \leq A_1 \otimes x\} = \{x : x = T \otimes u, u \in \mathbb{R}_{\mathbb{0}}^2\}$.

4.2 Algebraic expression of all the solutions of $\mathcal{A}_-$

First, let us define the fundamental notion in the following

Definition 4.1 (Free variable). *The variable x_i, $i \in \{1,\ldots,n\}$ is said to be free for the whole SMA $\mathcal{A}$ if x_i can take any value in $\mathbb{R}_{\mathbb{O}}$.*

It is clear that if the SMA $\mathcal{A}$ admits at least one free variable then the SMA cannot be reduced to the trivial solution $x = (\mathbb{O})$.

For the strictly negative part of the SMA $\mathcal{A}$ denoted $\mathcal{A}_-$ we have the following useful characterization.

Proposition 4.1 (Characterization of a Free variable in $\mathcal{A}_-$). *The variable x_i, $i \in \{1,\ldots,n\}$ is said to be free if $\forall l = 1,\ldots,\ell$: $a_i^l = e_i$, recalling that e_i is the ith row vector of identity matrix I_n and also the ith vector of the canonical basis of $\mathbb{R}_{\mathbb{O}}^n$.*

This characterization is not relevant for the positive SMA $\mathcal{A}_+$ because $x = (\mathbb{1})$ is always a non trivial solution for $\mathcal{A}_+$. However, if the set of free variables for $\mathcal{A}_-$ is empty in the sense of Proposition 4.1 then $\forall l = 1,\ldots,\ell \; \exists \lim_{k\to+\infty} A_l^{\otimes(k)} = O$ because by construction A_l verifies condition (7). And $x = (\mathbb{O})$ is the only solution of $\mathcal{A}_-$ thus to the whole SMA $\mathcal{A}$. Otherwise, because we assume **(W)** true we can assume that $\exists k : 1 \le k < n$ such that the variables $x_1,\ldots,x_k$ are free. In this case, by definition of a free variable all the matrices have the same structure which is:

$$\forall l = 1,\ldots,\ell: \; A_l = \begin{pmatrix} I_k & O_{k,n-k} \\ B_l & C_l \end{pmatrix}. \tag{12}$$

The condition (7) implies that the $(n-k)\times k$-matrix B_l and the $(n-k)\times(n-k)$-matrix C_l have all their entries $< \mathbb{1}$.

To obtain all the solutions as $(\max,+)$-linear functions of the free variables $x_1,\ldots,x_k$ we apply Result 2.1 using the following **main decomposition** defined by:

$$x \le A_l \otimes x \Leftrightarrow x \le A \otimes x \oplus b, \tag{13a}$$

with matrix A and vector b defined by (index l omitted for A and b):

$$A = \begin{pmatrix} O_{k,k} & O_{k,n-k} \\ O_{n-k,k} & C_l \end{pmatrix}. \tag{13b}$$

and

$$b = \begin{pmatrix} I_k \\ B_l \end{pmatrix} \otimes u, \; u = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}. \tag{13c}$$

And we obtain the expression of the greatest element of the set $\mathcal{S}(A,b)$ which is $x = A^* \otimes b$. Noticing that: $A^* = \begin{pmatrix} I_k & O_{k,n-k} \\ O_{n-k,k} & C_l^* \end{pmatrix}$, we have: $x = \begin{pmatrix} I_k \\ C_l^* \otimes B_l \end{pmatrix} \otimes u$. Denoting $J_l = C_l^* \otimes B$ and $T_l = \begin{pmatrix} I_k \\ J_l \end{pmatrix}$ we have the following set equality:

$$\{x : x \le A_l \otimes x\} = \{x : x = T_l \otimes u, u \in \mathbb{R}_{\mathbb{O}}^k\}, \tag{14}$$

for all $l = 1,\ldots,\ell$.

Noticing that $\mathcal{A}_- = \cap_{l=1}^{l}\{x : x \le A_l \otimes x\}$ we want to find a matrix T such that $\mathcal{A}_- = \{x : x = T \otimes u, u \in \mathbb{R}_{\mathbb{O}}^k\}$. Because the variables of the vector u are free T has necessarily the form

$T = \begin{pmatrix} I_k \\ X \end{pmatrix}$ where X is a $(n-k) \times k$-matrix to be computed as the solution of the followings ℓ systems of inequalities: $\forall l = 1, \ldots, \ell$:

$$T \leq A_l \otimes T \Leftrightarrow \left\{ \begin{array}{lll} I_k & \leq & I_k \\ X & \leq & B_l \oplus C_l \otimes X. \end{array} \right.$$

And the inequality $X \leq B_l \oplus C_l \otimes X$ is equivalent to $X \leq C_l^* \otimes B_l = J_l$. In other words we have $T \leq T_l$, $\forall l = 1, \ldots, \ell$. Thus, the greatest matrix satisfying all the previous inequalities is the matrix: $T^\wedge := T_1 \wedge \cdots \wedge T_\ell$ which can be rewritten as:

$$T^\wedge = \begin{pmatrix} I_k \\ J^\wedge \end{pmatrix}, \tag{15a}$$

with the matrix $J^\wedge$ defined by:

$$J^\wedge = \bigwedge_{l=1}^{\ell} C_l^* \otimes B_l. \tag{15b}$$

In conclusion, we have the following set equalities:

$$\mathcal{A}_- = \{x : x \leq A_1 \otimes x \wedge \cdots \wedge A_\ell \otimes x\} = \{x : x = T^\wedge \otimes u, u \in \mathbb{R}_{\mathbb{0}}^k\}. \tag{16}$$

recalling that the matrix $T^\wedge$ is defined by (15a)-(15b).

5 Looking for solutions of $\mathcal{A}_+$ as functions of the free variables of $\mathcal{A}_-$

The SMA $\mathcal{A}_+$ is associated with the inequality $x \leq A_{\ell+1} \otimes x \wedge \ldots \wedge A_L \otimes x$. Writing $x = T^\wedge \otimes u = t_1^\wedge \otimes x_1 \oplus \cdots \oplus t_k^\wedge \otimes x_k$, where $t_1^\wedge, \ldots, t_k^\wedge$ are the k column vectors of the matrix $T^\wedge$. We have the following characterization of a free variable in $\mathcal{A}_+$.

Proposition 5.1 (Characterization of a free variable in $\mathcal{A}_+$)**.** *A variable x_i is free for $\mathcal{A}_+$ if x_i is free for $\mathcal{A}_-$, thus necessarily $i \in \{1, \ldots, k\}$, and:*

$$\forall l = \ell + 1, \ldots, L : t_i^\wedge \leq A_l \otimes t_i^\wedge. \tag{17}$$

Let us note that for the SMA $\mathcal{A}_-$ we already have:

$$\forall l = 1, \ldots, \ell, \forall i = 1, \ldots, k : t_i^\wedge \leq A_l \otimes t_i^\wedge,$$

by definition of the matrix $T^\wedge$. We then have the following cases.

5.1 The two trivial cases

The two obvious cases are the following ones.
CASE 1. $\forall i = 1, \ldots, k$, the system of inequalities (17), Proposition 5.1, is not verified.
Thus, the free variables of $\mathcal{A}_-$, ie $\{x_1, \ldots, x_k\}$ are not free variables in $\mathcal{A}_+$. And then $x = (\mathbb{0})$ is the only solution of the whole SMA $\mathcal{A}$. We are able to conclude that MAP is no.
CASE 2. $\forall i = 1, \ldots, k$, the system of inequalities (17), Proposition 5.1, is verified.
Thus, all the free variables of $\mathcal{A}_-$, ie $\{x_1, \ldots, x_k\}$, are also free variables in $\mathcal{A}_+$. And then the whole SMA is defined by:

$$\mathcal{A} = \{x : A_1 \otimes x \wedge \ldots \wedge A_L \otimes x\} = \{x : x = T^\wedge \otimes u, u \in \mathbb{R}_{\mathbb{O}}^k\}, \tag{18}$$

recalling that the matrix $T^\wedge$ is defined by (15a)-(15b).

5.2 Otherwise, there exist $1 \leq k' < k$ free variables for $\mathcal{A}_+$

W. l. o. g. we can assume that the first k' variables, ie $\{x_1, \ldots, x_{k'}\}$ are still free. Let us define the three following vectors of variables $u^1 = \begin{pmatrix} x_1 \\ \vdots \\ x_{k'} \end{pmatrix}$, $\overline{u}^1 = \begin{pmatrix} x_{k'+1} \\ \vdots \\ x_k \end{pmatrix}$ and $\overline{u} = \begin{pmatrix} x_{k+1} \\ \vdots \\ x_n \end{pmatrix}$. It is clear that if we force $\overline{u}^1 = (\mathbb{O})$ which means that $u = \begin{pmatrix} u^1 \\ \mathbb{O} \end{pmatrix}$ then $x = T^\wedge \otimes u$ is a solution of the whole SMA $\mathcal{A}$ and MAP is yes in strongly polynomial time. But hereafter we show that it is possible to go a little bit further.

Noticing that the matrix $J^\wedge$ has been obtained by setting $\overline{u} = X \otimes u$, $T = \begin{pmatrix} I_k \\ X \end{pmatrix}$ and then solving $T \leq A_l \otimes T$, $l = 1, \ldots, \ell$. We do the same here with $\overline{u}^1$ and u^1. Thus, let us set:

$$\overline{u}^1 = F \otimes u^1,$$

where F is the $(k - k') \times k'$-unknown matrix which has to be calculated. Then, we have $u = \begin{pmatrix} u^1 \\ \overline{u}^1 \end{pmatrix} = T^1 \otimes u^1$, with:

$$T^1 = \begin{pmatrix} I_{k'} \\ F \end{pmatrix}. \tag{19}$$

The state space $\mathbb{R}_{\mathbb{O}}^n$ is now partitioned according to the vectors $(u^1, \overline{u}^1, \overline{u})$.

According to the new partition of the sate space the $n \times k$-matrix $T^\wedge$ is rewritten as follows:

$$T^\wedge = \begin{pmatrix} I_{k'} & O_{k',k-k'} \\ O_{k-k',k'} & I_{k-k'} \\ J & K \end{pmatrix}. \tag{20}$$

Now let us denote A any $n \times n$-matrix element of the set $\{A_{\ell+1}, \ldots, A_L\}$. According to the new partition of the state space we write:

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{pmatrix}.$$

Let us express all as function of the vector of the free variables u^1. It comes:

$$u = T^1 \otimes u^1,\ x = T^\wedge \otimes u = T^\wedge \otimes T^1 \otimes u^1,$$

with the product

$$T^\wedge \otimes T^1 = \begin{pmatrix} I_{k'} \\ F \\ J \oplus K \otimes F \end{pmatrix}.$$

which could be rewritten as:

$$T^{\wedge} \otimes T^{1} = \begin{pmatrix} I_{k'} \\ O_{k-k',k'} \\ J \end{pmatrix} \oplus \begin{pmatrix} O_{k',k'} \\ I_{k-k'} \\ K \end{pmatrix} \otimes F. \tag{21}$$

And from (21) we have the following expression for $A \otimes T^{\wedge} \otimes T^{1}$:

$$A \otimes T^{\wedge} \otimes T^{1} = \begin{pmatrix} A_{1,1} \oplus A_{1,3} \otimes J \\ A_{2,1} \oplus A_{2,3} \otimes J \\ A_{3,1} \oplus A_{3,3} \otimes J \end{pmatrix} \oplus \begin{pmatrix} A_{1,2} \oplus A_{1,3} \otimes K \\ A_{2,2} \oplus A_{2,3} \otimes K \\ A_{3,2} \oplus A_{3,3} \otimes K \end{pmatrix} \otimes F. \tag{22}$$

The inequality $x \leq A \otimes x$ is expressed as the following inequality between functions of u^1: $T^{\wedge} \otimes T^{1} \otimes u^{1} \leq A \otimes T^{\wedge} \otimes T^{1} \otimes u^{1}$. Because u^1 is a vector of free variables we have to solve the following matrix inequality:

$$T^{\wedge} \otimes T^{1} \leq A \otimes T^{\wedge} \otimes T^{1},$$

developing the product of matrices we obtain:

$$\underbrace{\begin{pmatrix} I_{k'} \\ O_{k-k',k'} \\ J \end{pmatrix}}_{u^1} \oplus \underbrace{\begin{pmatrix} O_{k',k'} \\ I_{k-k'} \\ K \end{pmatrix} \otimes F}_{\overline{u}^1} \leq \underbrace{\begin{pmatrix} A_{1,1} \oplus A_{1,3} \otimes J \\ A_{2,1} \oplus A_{2,3} \otimes J \\ A_{3,1} \oplus A_{3,3} \otimes J \end{pmatrix}}_{u^1} \oplus \underbrace{\begin{pmatrix} A_{1,2} \oplus A_{1,3} \otimes K \\ A_{2,2} \oplus A_{2,3} \otimes K \\ A_{3,2} \oplus A_{3,3} \otimes K \end{pmatrix} \otimes F}_{\overline{u}^1}. \tag{23}$$

Recalling that $\oplus$ is the max 'addition', inequality (23) is equivalent to:

$$\underbrace{\begin{pmatrix} I_{k'} \\ O_{k-k',k'} \\ J \end{pmatrix}}_{u^1} \leq \underbrace{\begin{pmatrix} A_{1,1} \oplus A_{1,3} \otimes J \\ A_{2,1} \oplus A_{2,3} \otimes J \\ A_{3,1} \oplus A_{3,3} \otimes J \end{pmatrix}}_{u^1} \oplus \underbrace{\begin{pmatrix} A_{1,2} \oplus A_{1,3} \otimes K \\ A_{2,2} \oplus A_{2,3} \otimes K \\ A_{3,2} \oplus A_{3,3} \otimes K \end{pmatrix} \otimes F}_{\overline{u}^1}. \tag{24a}$$

And

$$\underbrace{\begin{pmatrix} O_{k',k'} \\ I_{k-k'} \\ K \end{pmatrix} \otimes F}_{\overline{u}^1} \leq \underbrace{\begin{pmatrix} A_{1,1} \oplus A_{1,3} \otimes J \\ A_{2,1} \oplus A_{2,3} \otimes J \\ A_{3,1} \oplus A_{3,3} \otimes J \end{pmatrix}}_{u^1} \oplus \underbrace{\begin{pmatrix} A_{1,2} \oplus A_{1,3} \otimes K \\ A_{2,2} \oplus A_{2,3} \otimes K \\ A_{3,2} \oplus A_{3,3} \otimes K \end{pmatrix} \otimes F}_{\overline{u}^1}. \tag{24b}$$

Because u^1 is the vector of the free variables for the SMA then in particular $\forall A \in \{A_{\ell+1}, \ldots, A_L\}$ we have the following inequality:

$$\underbrace{\begin{pmatrix} I_{k'} \\ O_{k-k',k'} \\ J \end{pmatrix}}_{u^1} \leq \underbrace{\begin{pmatrix} A_{1,1} \oplus A_{1,3} \otimes J \\ A_{2,1} \oplus A_{2,3} \otimes J \\ A_{3,1} \oplus A_{3,3} \otimes J \end{pmatrix}}_{u^1},$$

which is true. Thus, the inequality (24a) holds $\forall F$. And the unknown matrix F will be calculated using only inequality (24b). We introduce the following new matrices Z and B:

$$Z = \begin{pmatrix} I_{k-k'} \\ K \end{pmatrix} \otimes F, \tag{25}$$

and
$$B = \left(\begin{array}{c} A_{2,1} \oplus A_{2,3} \otimes J \\ A_{3,1} \oplus A_{3,3} \otimes J \end{array} \right). \tag{26}$$

Noticing that:
$$\left(\begin{array}{c} A_{2,2} \oplus A_{2,3} \otimes K \\ A_{3,2} \oplus A_{3,3} \otimes K \end{array} \right) \otimes F = C \otimes \left(\begin{array}{c} I_{k-k'} \\ K \end{array} \right) \otimes F = C \otimes Z,$$
where C is the following $(n-k') \times (n-k')$-matrix:
$$C = \left(\begin{array}{cc} A_{2,2} & A_{2,3} \\ A_{3,2} & A_{3,3} \end{array} \right). \tag{27}$$

The inequality (24b) is equivalent to:
$$Z \leq C \otimes Z \oplus B \tag{28a}$$
and
$$\left(\begin{array}{c} I_{k-k'} \\ K \end{array} \right) \otimes F = Z. \tag{28b}$$

Then we solve system (28a)-(28b) in this order as follows.
a/ Solving inequality (28a). We have two cases.

- Case 1. C^* exists which means eg that in the valued graph $G(C)$ associated with the matrix C there does not exist elementary circuit with weight $> \mathbb{1}$). In this case the greatest solution of (28a) is $Z = C^* \otimes B$.
- Case 2. C^* does not exist. Then, just take $Z = B$ which verifies obviously: $B \leq C \otimes B \oplus B$.

Of course in Case 1 we have $B \leq Z$, by definition of the Kleene star and B is always a solution of (28a).
b/ Solving (28b) knowing the expression of Z.
Residuation theory (see (1), (2), (3) and (4)) provides us the greatest (in the sense of the partial order $\leq$ between matrices) solution for the matrix F which satifies the inequality:
$$\left(\begin{array}{c} I_{k-k'} \\ K \end{array} \right) \otimes F \leq Z \tag{29}$$
which will be denoted:
$$F(A) = \left(\begin{array}{c} I_{k-k'} \\ K \end{array} \right) \backslash Z. \tag{30}$$
Recalling that A denotes any matrix of the set of matrices $\{A_{\ell+1}, \ldots, A_L\}$.

Using the same method we get the algebraic formulae for the matrices $F(A_l)$, $l = \ell+1, \ldots, L$. Moreover, noticing that by definition of the residuate the inequality (29) is true for all matrices F such that $F \leq F(A)$. Thus, we get the expression of the greatest matrix which satisfies all the constraints of the form (29), say $F^{\wedge}$, defined by:
$$F^{\wedge} := F(A_{\ell+1}) \wedge \ldots \wedge F(A_L). \tag{31}$$

And finaly, we obtain an algebraic formula for all the solutions of the whole SMA $\mathcal{A}$ given by:
$$\mathcal{A} = \{x \in \mathbb{R}^n_{\mathbb{0}} : x = T^{\wedge} \otimes \left(\begin{array}{c} I_{k'} \\ F \end{array} \right) \otimes u^1, u^1 \in \mathbb{R}^{k'}_{\mathbb{0}}, F \leq F^{\wedge}\}. \tag{32}$$

6 Time Complexity in $\mathcal{O}(mn^3)$

Let us recall that we consider SMA $\mathcal{A}$ with m max-atoms and n variables $V = \{x_1, \ldots, x_n\}$. And that if hypothesis **(W)** is false answer to question MAP is yes in $\mathcal{O}(1)$, indeed $x = (\mathbb{1})$ is a non trivial solution of $\mathcal{A}$. When **(W)** is true the present paper treats the question MAPall.

Due to condition (7) the SMA $\mathcal{A}$ is divided into two SMA. The strictly negative SMA $\mathcal{A}_-$ associated with the $n \times n$-mtrices $[A_1, \ldots, A_\ell]$, and the positive SMA $\mathcal{A}_+$ associated the $n \times n$-mtrices $[A_{\ell+1}, \ldots, A_L]$, where the integers ℓ and L respectively defined by (10a) and (10b).

The greatest complexity for finding all the solutions of the SMA $\mathcal{A}_-$ (see subsection 4.2) concerns the computation of the Kleene star of a $n \times n$-matrix which is $\mathcal{O}(n^3)$. And there are ℓ matrices.

Fort the positive part, ie SMA $\mathcal{A}_+$, the approach developed in subsection 5.2 requires to detect if the Kleene star of the $(n-k') \times (n-k')$ matrix C defined by (27) exists. Which means that its valued graph $G(C)$ does not contain elemntary circuits with weight $> \mathbb{1}$. The detection of such circuit can be done using eg the Bellman-Ford algorithm which complexity is here at most $\mathcal{O}(n^3)$, indeed $n - k' \leq n$. And the positive part has $L - \ell$ matrices C to analyze. Thus time complexity is about $\mathcal{O}(\ell n^3) + \mathcal{O}((L-\ell)n^3) \approx \mathcal{O}(mn^3)$ (because L is $\leq m$ by definition). And the space complexity is about mn^2 because we have to store L $n \times n$-matrices.

7 Back to numerical examples of subsection 1.1

Hereafter, we solve the three problems mentioned in subsection 1.1.

7.1 SMA S + MAP as a model of an hypergraph problem P5

The particular hypergraph problem (see Figure 1) is (PTIME) equivalent to the SMA S + MAP. Using results of subsection 3.2 S is equivalent to the inequality $x \leq A_1 \otimes x \wedge A_2 \otimes x$ with $x = (x_i)_{i=1}^4$. The strictly negative part of S, say S_-, is associated with the matrix A_1 defined by:

$$A_1 = \begin{pmatrix} I_2 & O_{2,2} \\ B_1 & C_1 \end{pmatrix}, \; B_1 = \begin{pmatrix} 10^{\otimes(-1)} & \mathbb{0} \\ \mathbb{0} & 1^{\otimes(-1)} \end{pmatrix}, \; C_1 = \begin{pmatrix} \mathbb{0} & \mathbb{0} \\ 1^{\otimes(-1)} & \mathbb{0} \end{pmatrix}. \tag{33}$$

And the positive part of S is associated with the matrix $A_2 = \begin{pmatrix} \mathbb{1} & \mathbb{0} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & \mathbb{1} \\ \mathbb{0} & \mathbb{0} & \mathbb{1} & \mathbb{0} \\ \mathbb{0} & 25 & 25 & \mathbb{0} \end{pmatrix}$.

By Proposition 4.1 the vector $u = (x_i)_{i=1}^2$ is the column vector of the free variables in S_-. Applying results of subsection 4.2 we have the following set equalities:

$$S_- = \{x : x \leq A_1 \otimes x\} = \{x = T^\wedge \otimes u, u \in \mathbb{R}^2_{\mathbb{0}}\}.$$

Where the matrix $T^\wedge$ is defined by (15a)-(15b). Specifying the matrix $J^\wedge$ defined by (15b): $J^\wedge = C_1^* \otimes B_1 = \begin{pmatrix} 10^{\otimes(-1)} & \mathbb{0} \\ 11^{\otimes(-1)} & 1^{\otimes(-1)} \end{pmatrix}$, the matrix $T^\wedge$ has the following numerical entries:

$$T^\wedge = \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & \mathbb{1} \\ 10^{\otimes(-1)} & \mathbb{0} \\ 11^{\otimes(-1)} & 1^{\otimes(-1)} \end{pmatrix}. \tag{34}$$

And the reader can check that $t_i^\wedge \leq A_2 \otimes t_i^\wedge$ is true for $i=1$ thus MAP is yes and the hypergraph problem is also solved. Note that the vector $t_1^\wedge$ is a solution of S but we do not exactly retrieve the numerical value obtained in [4], Example 1 which is $x^b = \begin{pmatrix} \alpha(u) = \mathbb{1} \\ \alpha(y) = 11^{\otimes(-1)} \\ \alpha(x) = 10^{\otimes(-1)} \\ \alpha(z) = 11^{\otimes(-1)} \end{pmatrix}$ because we do not explore all the solutions of S. However, the reader can check that: $x^b = T^\wedge \otimes \begin{pmatrix} \mathbb{1} \\ 11^{\otimes(-1)} \end{pmatrix}$.

7.2 Solving SMA S', subsection 1.1

The SMA S' only differs from the SMA S by its positive part which is associated with the matrix $A_2' = \begin{pmatrix} \mathbb{0} & 9 & 9 & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & \mathbb{1} \\ \mathbb{0} & \mathbb{0} & \mathbb{1} & \mathbb{0} \\ \mathbb{0} & 25 & 25 & \mathbb{0} \end{pmatrix}$.

Then it is easy to check that none of the inequalities $t_i^\wedge \leq A_2 \otimes t_i^\wedge$, $i = 1, 2$ is verified. Thus, $x = (\mathbb{0})$ is the only solution and we retrieve the result of [4], Example 2.

7.3 Looking for all solutions of the SMA S'', subsection 1.1

Once again the only changes concern the positive part of the SMA S'' associated with the matrix $A_2'' = \begin{pmatrix} \mathbb{0} & 12 & 12 & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & \mathbb{1} \\ \mathbb{0} & \mathbb{0} & \mathbb{1} & \mathbb{0} \\ \mathbb{0} & 25 & 25 & \mathbb{0} \end{pmatrix}$. Here, inequality $t_i^\wedge \leq A_2' \otimes t_i^\wedge$ is true for $i = 1$ and false for $i = 2$. Thus, at this step MAP is yes. But we are now looking for all solutions of the SMA S'' using results and notations of subsection 5.2. First, let us define the new partition of the state space $\mathbb{R}_{\mathbb{0}}^4$ by $u^1 = (x_1)$, $\overline{u}^1 = (x_2)$ and $\overline{u} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$. And set: $\overline{u}^1 = F \otimes u^1$ where $F = (f)$, f is the unknown scalar to be calculated.

The matrix $T^\wedge$ defined by (34) is rewritten as follows:

$$T^\wedge = \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & \mathbb{1} \\ J & K \end{pmatrix}, \; J = \begin{pmatrix} 10^{\otimes(-1)} \\ 11^{\otimes(-1)} \end{pmatrix}, \; K = \begin{pmatrix} \mathbb{0} \\ 1^{\otimes(-1)} \end{pmatrix}.$$

And the matrix A_2'' is rewritted as:

$$A_2'' = \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{pmatrix},$$

with:
$A_{1,1} = \begin{pmatrix} \mathbb{0} \end{pmatrix}$, $A_{1,2} = \begin{pmatrix} 12 \end{pmatrix}$, $A_{1,3} = \begin{pmatrix} 12 & \mathbb{0} \end{pmatrix}$, $A_{2,1} = \begin{pmatrix} \mathbb{0} \end{pmatrix}$, $A_{2,2} = \begin{pmatrix} \mathbb{0} \end{pmatrix}$, $A_{2,3} = \begin{pmatrix} \mathbb{0} & \mathbb{1} \end{pmatrix}$, $A_{3,1} = \begin{pmatrix} \mathbb{0} \\ \mathbb{0} \end{pmatrix}$, $A_{3,2} = \begin{pmatrix} \mathbb{0} \\ 25 \end{pmatrix}$, $A_{3,3} = \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ 25 & \mathbb{0} \end{pmatrix}$.

We have to solve system (28a)-(28b) specified here by: $F = (f)$, $Z = t \otimes F$, $t = \begin{pmatrix} \mathbb{1} \\ \mathbb{0} \\ 1^{\otimes(-1)} \end{pmatrix}$ (cf. (25)), $B = \begin{pmatrix} 11^{\otimes(-1)} \\ 10^{\otimes(-1)} \\ 15 \end{pmatrix}$ (cf. (26)), $C = \begin{pmatrix} \mathbb{0} & \mathbb{0} & \mathbb{1} \\ \mathbb{0} & \mathbb{1} & \mathbb{0} \\ 25 & 25 & \mathbb{0} \end{pmatrix}$ (cf. (27)).

In our case C^* does not exist. So, Case 2 applies to solve (28a), which means that we have to solve: $Z = B$, where $Z = t \otimes F$. Residuation theory (see (1), (2), (3) and (4)) provides the upper bound on scalar f: $f^\wedge = t \backslash B = 11^{\otimes(-1)}$, (see the general definition based on (30) and (31)).

Finally, the set of all the solutions of S'' is deduced from application of result (32) and:

$$S'' = \{x \in \mathbb{R}^4_{\mathbb{0}}; x = T^\wedge \otimes \begin{pmatrix} \mathbb{1} \\ f \end{pmatrix} \otimes x_1, x_1 \in \mathbb{R}_{\mathbb{0}}, f \le f^\wedge = 11^{\otimes(-1)}\}, \tag{35}$$

recalling that the matrix $T^\wedge$ is specified here by (34).

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